Turbomachinery & Turbulence.
Lecture 3: Similitude.

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March 16, 2015

Dimensional analysis

For a given machine of characteristic size R, the quantities of interest are:

- The variation of specific mechanical energy of the fluid $\Delta e_{mech} = \Delta \left(\frac{P}{\rho} + \frac{C^2}{2} + gz \right)$, or the "Head" $gH = \Delta e_{mech}$.
- The shaft power that is exchanged P.
- The efficiency η.

As a function of:

- The rotation rate ω.
- The volume flow-rate Q.
- The fluid properties ρ, μ.
- The dimensionless geometry.

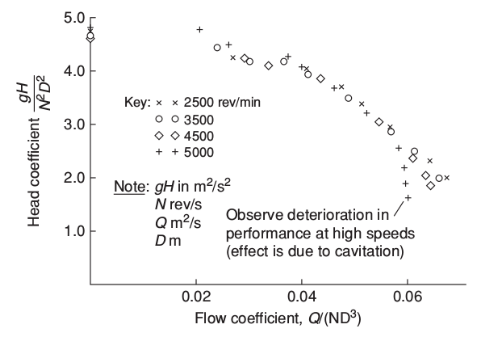
Dimensional analysis

- Three primary dimensions (M, L, T).
- Five independent variables (R, ω, Q, ρ, μ).
- Three dependent variables (gH, P, η).
- How many dimensionless groups ?
- What is the customary choice?

Usual choice

- Primary dimensions based on R, ω, ρ.
- Dimensionless independent variables:
  - $Phi = \frac{Q}{\omega R^3}$: flow coefficient.
  - $Re = \frac{\rho \omega R^2}{\mu}$: Reynolds number.
- Dimensionless dependent variables:
  - $\Psi = \frac{gH}{\omega^2 R^2}$: Head coefficient.
  - $\Pi = \frac{P}{\rho \omega^3 R^5}$: power coefficient.
  - $\eta = \frac{\Phi\Psi}{\Pi}$ (pump) or $\eta = \frac{\Pi}{\Phi\Psi}$ (turbine).
- A turbomachine *characteristic* is the graphical representation of $\Psi = f(\Phi, Re)$, and $\eta = g(\Phi, Re)$.

Example: pump characteristic

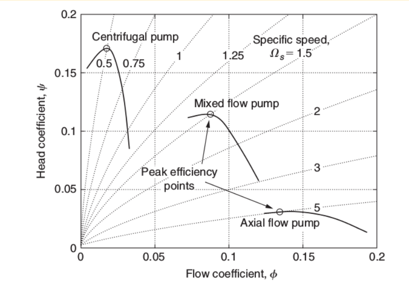


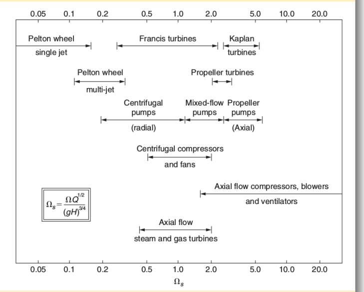
- For $Re \gtrsim 2 \times 10^5$, it is experimentally found that $\Psi = f(\Phi)$ is independent of Re.
- η has a very small dependence ($+1\%$ for $Re \times 10$).

Specific speed

- At high *Re*, no Reynolds-dependence.
- At the point of maximum efficiency: $\Phi = \mathrm{cte}$, $\Psi = \mathrm{cte}$.
- Combination of Φ, Ψ to eliminate R, specific speed Ω:

$$\Omega = \frac{\Phi^{\frac{1}{2}}}{\Psi^{\frac{3}{4}}} = \frac{\omega\sqrt{Q}}{(gH)^{\frac{3}{4}}}$$

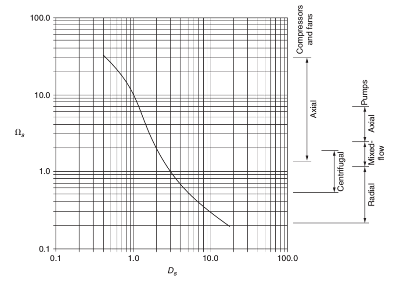


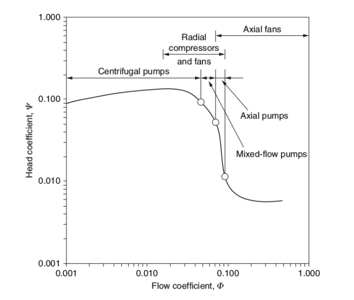


Specific radius

- Combination of Φ, Ψ to eliminate ω, specific radius Λ:

$$\Lambda = \frac{\Psi^{\frac{1}{4}}}{\Phi^{\frac{1}{2}}} = \frac{R\left(gH\right)^{\frac{1}{4}}}{\omega\sqrt{Q}}$$





Exercice 1

- A hydraulic turbine with a runner outside diameter of 4.31 m operates with an effective head of $H = 543$ m at a volume flow rate of $Q = 71.5$ m^3.s^{-1} and produces 350 MW of shaft power at a rotational speed of 333 rpm.
- Determine the specific speed, the specific diameter, and efficiency of this turbine.
- Another geometrically and dynamically similar turbine with a runner 6.0 m diameter is to be built to operate with an effective head of $H = 500$ m. Determine the required flow rate, the expected power output, and the rotational speed of the turbine.

Dimensional analysis

We assume a fluid that obeys the ideal perfect gas law.
For a given machine of characteristic size R, the quantities of interest are:

- The variation of specific mechanical energy of the fluid Δe_{mech}, *i.e.* the *isentropic* total enthalpy variation $\Delta h_{0,s} = \Delta e_{mech}$.
- The shaft power that is exchanged P, *or* the total enthalpy variation $P = \dot{m}\Delta h_0$.
- The efficiency η.

As a function of:

- The rotation rate ω.
- The mass flow-rate $\dot{m}$.
- The stagnation density at inlet $\rho_{0,1}$.
- The fluid viscosity μ.
- The dimensionless geometry.
- The ratio of specific heats $\gamma = \frac{c_p}{c_v}$.
- The fluid constant r.
- The stagnation pressure at inlet $p_{0,1}$.
- The stagnation temperature at inlet $T_{0,1}$.
- The stagnation sound celerity at inlet $a_{0,1} = \sqrt{\gamma r T_{0,1}}$.

Dimensional analysis

- Three primary dimensions (M, L, T).
- Seven independent variables (R, ω, $\dot{m}$, $\rho_{0,1}$, μ, $a_{0,1}$, γ).
- Three dependent variables ($\Delta h_{0,s}$, P, η).

Dimensional analysis, first set of selected variables

- Primary dimensions based on R, ω, $\rho_{0,1}$.
- Dimensionless independent variables:
  - $\Phi = \frac{\dot{m}}{\rho_{0,1}\omega R^3}$: flow coefficient.
  - $Re = \frac{\rho_{0,1}\omega R^2}{\mu}$: Reynolds number.
  - $M_{tip} = \frac{\omega R}{a_{0,1}}$: tip Mach number.
  - γ.
- Dimensionless dependent variables:
  - $\Pi_1 = \frac{\Delta h_{0,s}}{\omega^2 R^2}$.
  - $\Pi_2 = \frac{P}{\rho_{0,1}\omega^3 R^5}$.
  - $\eta = \frac{\Phi\Pi_1}{\Pi_2}$ (compressor).
- Finally:

$$\Pi_1, \Pi_2, \eta = f\left(\Phi, Re, M_{tip}, \gamma\right)$$

Dimensional analysis, second set of selected variables

- Primary dimensions based on R, $a_{0,1}$, $\rho_{0,1}$.
- Dimensionless independent variables:
  - $\widehat{m} = \frac{\dot{m}}{\rho_{0,1} a_{0,1} R^2}$: discharge/ meridional Mach number.
  - $Re = \frac{\rho_{0,1} a_{0,1} R}{\mu}$: (weird) Reynolds number.
  - $M_{tip} = \frac{\omega R}{a_{0,1}}$: tip Mach number.
  - γ.
- Dimensionless dependent variables:
  - $\Pi_1 = \frac{\Delta h_{0,s}}{a_{0,1}^2}$.
  - $\Pi_2 = \frac{P}{\rho_{0,1} a_{0,1}^3 R^2}$.
  - $\eta = \frac{\widehat{m}\Pi_1}{\Pi_2}$.

Second set of selected variables, alternative form

- Use of the state law and of the isentropic relations.
- Isentropic stagnation enthalpy rise: $\Delta h_{0,s} = c_p(T_{0,2s} - T_{0,1})$.
- Isentropic relationship between temperature and pressure:

$$\frac{T_{0,2s}}{T_{0,1}} = \left(\frac{p_{0,2}}{p_{0,1}}\right)^{\frac{\gamma-1}{\gamma}}$$

- Thus:

$$\Delta h_{0,s} = c_p T_{0,1}\left[\left(\frac{p_{0,2}}{p_{0,1}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

- Since $a_{0,1}^2 = (\gamma - 1)c_p T_{0,1}$:

$$\Pi_1 = \frac{1}{\gamma - 1}\left[\left(\frac{p_{0,2}}{p_{0,1}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = f\left(\frac{p_{0,2}}{p_{0,1}}, \gamma\right)$$

Second set of selected variables, alternative form

- For "$\widehat{m}$".

$$\widehat{m} = \frac{\dot{m}\sqrt{\gamma r T_{0,1}}}{\gamma R^2 p_{0,1}}$$

- For "Π_2":

$$\Pi_2 = \frac{\dot{m} c_p \Delta T_0}{\rho_{0,1} a_{0,1}^2 R^2 a_{0,1}} = \frac{\widehat{m}}{\gamma - 1} \frac{\Delta T_0}{T_{0,1}}$$

- Finally:

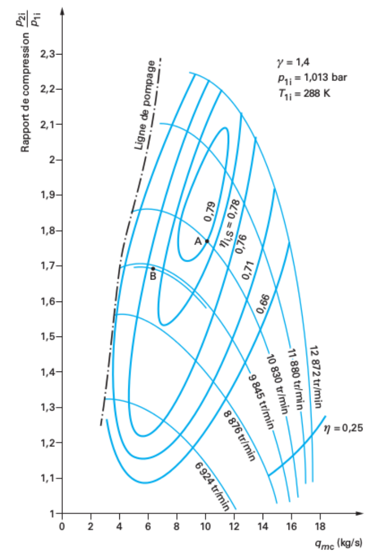
$$\frac{p_{0,2}}{p_{0,1}}, \eta, \frac{\Delta T_0}{T_{0,1}} = f\left(\frac{\dot{m}\sqrt{\gamma r T_{0,1}}}{\gamma R^2 p_{0,1}}, M_{tip}, \gamma\right)$$

$$\frac{p_{0,2}}{p_{0,1}} = \left(1 + \eta \frac{\Delta T_0}{T_{0,1}}\right)^{\frac{\gamma}{\gamma - 1}}$$

Exercice 1

- An air turbine is required for a dentist's drill. For the drill bit to effectively abrade tooth enamel, the turbine must rotate at high speed, around 350000 rpm. The turbine must also be very small so that it can be used to access all parts of a patient's mouth and an exit air flow rate in the region of 10 L/min is required for this. The turbine is to be driven by supply air at a pressure of 3 bar and a temperature of 300 K.
- Calculate the specific speed of the turbine and use this to determine the type of machine required. Also estimate the power consumption of the turbine.

Exercice 2



- Here is the characteristics of a centrifugal compressor, operating at standard conditions ($T0, 1 = 288$ K, $p_{0,1} = 1013$ hPa.
- Suppose there are two compressors in serial (on the same shaft). The first one is supplied with $\dot{m} = 10$ kg.s^{-1} and rotates at 10830 rpm (pressure ratio is 1.773, efficiency is 79%, point A).
- What is the working point of the second?
- what is the total pressure ratio and power consumption?