Dimensional Analysis and Similarity
How to prepare a fluid flow experiment ?

A. Danlos, F. Ravelet

DynFluid Laboratory, Arts et Métiers ParisTech,
151 Boulevard de l'Hôpital, 75013 Paris

January 23, 2014

This lecture deals with the experimental methods for fluid flows. The aim of this presentation is to study how to prepare a fluid flow experiment.

Introduction

In 1915, Lord Rayleigh suggests to think before performing complex and expensive experiments. Indeed, in his well-known "Principle of similitude" published in Nature, we can read this paragraph:

"I have often been impressed by the scanty attention paid even by original workers in physics to the great principle of similitude. It happens not infrequently that results in the form of laws are put forward as novelties on the basics of elaborate experiments, which might have been predicted a priori after a few minutes' consideration".

So it is necessary to learn thinking to be efficient in flows studies.

The Scuderia Ferrari Team is a good example to illustrate this basic principle. From April to October in 2011, this team encounter many problems due to an error of adjustement of Maranello Wind Tunnel, in which formula one cars are tested. The 150^o Italia Ferrari was lagged aerodynamically due to a problem with the calibration of the wind tunnel. Indeed, a scale changing in aerodynamic tests (from 50% to 60%) results on a different behaviour on track than in the wind tunnel. Renault encounters same problems in 2007. As there is only one wind tunnel for models real size, it is necessary to improve these models with scale changing. The question that may arise is "how to design a model in order to validate results and to understand what happens at full-scale?". This is the major problem of fluid flow experiments.

So, why is dimensional analysis crucial for setting up experiments?

- This method is used to analyze physical units. This point is the easier and the most known use of dimensional analysis.
- It is easy to create dimensionless numbers to generalize results (then results are validate not only for one experiment but also for one type of experiments).

- Dimensional analysis can simplify basic equations with the identification of negligible terms. It is often used with the balance equations in fluid mechanics.
- We can reduce relevant parameters needed too, for experimental, theoretical or numerical studies.
- It is a way to determine criteria to respect for model validation.

Some examples of the use of dimensional analysis and of similarity:

1. the **Viaduc de Millau** (Centre Scientifique et Technique du Bâtiment - CSTB): Real size $2460m \times 343m$ / Jules Verne wind tunnel size: 14m length and 6m width, with all climate conditions.
2. **La Fayette ship** (Bassin des Carênes, Val de Reuil): Real size 125m length, 25noeuds ($13m/s$ or $46km/h$) / Pool size: 545m, Models size: 10m length
3. the **Mont-Saint-Michel** (Sogreah Grenoble): Real size $387.47m^2$ / Pool size 40m length.
4. the **Nantes Stadium (CSTB)**: Real size $105m \times 68m$ / wind tunnel size: $14m \times 6m$
5. **Anti-oil boom** (LEPTAB La Rochelle): Real size 300m / Pool size 20m length, Models size 1/15
6. **A380** (ONERA, F1 wind tunnel, Toulouse): Real size $72.72m \times 79.75m \times 24.09m$, Speed $900km/h$ (Mach 0.88) / F1 Size $3.5m \times 4.5m$, Mach 0.36

These different examples will illustrate the interest of the dimensional analysis.

The outline of this lecture is then the following. First, we will remember the **International System of Units** and their importance. Next, we will study the **dimensional analysis** and its advantages. Finally we will look at the **similarity principle** in order to understand how to design fluid flow experiments.

1 International System of Units

The International System of Units, universally abbreviated SI (from the French Le Système International d'Unités), is the modern metric system of measurement. The value of a quantity is generally expressed as the product of a number and a unit. The unit is simply a particular example of the quantity concerned which is used as a reference, and the number is the ratio of the value of the quantity to the unit. For a particular quantity, many different units may be used. A unit is a particular physical quantity, defined and adopted by convention, with which other particular quantities of the same kind are compared to express their value. For a long time, all quantities are measured with local units, different in all countries but also in all cities. Each community uses its own standards. Scientists decided to determine reference units to exchange their knowledge and research and to facilitate trade and economy in countries. The International System of Units is founded on 7 base units for 7 base quantities assumed to be mutually independent.

It took time to create these 7 base units as we can see on the timeline. From the Revolution until 1971, scientists tried to agree about the name, and the accurate method to calculate standards of each unit. The first step to obtain this result is born in France, with the Acadmie des Sciences. Its members decided to use meter and the "grave", which was the first name of kilogram. Then the Decimal Metric System was improved in order to be use in all scientific domains as physics, electronics, but also in chemistry.

The International System SI is now used in most countries but not all! For example, as we can see on the mapping of the SI use, the United States still use their own standards, except in the scientific community.

Finally, the 7 base units are: **meter**, **kilogram**, **second**, **ampere**, **Kelvin**, **mole** and **candela**. Other quantities are derived quantities so they may be written in terms of the base quantities by the equations of physics. Each unit is linked to a measurable physical quantity which defines its kind and which is called dimension.

All dimensions of the derived quantities are written as products of powers of the dimensions of the base quantities by writing equations.

This property permit to validate results in equations for instance because **Inhomogeneous result is necessarily wrong!** But, it is important to remember that an homogeneous result is not perforce good....

For example: $x = \frac{1}{2}gt^2 + v_o t$ / unit=m

Homogeneity rules are rewritten here in order to clean the bases. So the dimensional analysis is a good method to remember units and to validate results. In order to illustrate this property, we can calculate the dynamic viscosity unit or the kinematic viscosity unit.

2 Dimensional Analysis

Now we are going to study the dimensional analysis and its applications.

Dimensional analysis is a method to **find physical laws** when a theoretical resolution is too complex. If we consider a physical law w which is a function of independant dimensional quantities x,y and z, we can write the physical law as a product: $w = Kx^{\alpha}y^{\beta}z^{\gamma}$. Here, K is a dimensionless constant called dimensionless number.

We can conclude that finding a physical lax is finding dimensionless numbers. This theory is the basis of the **Vaschy-Buckingham theorem**.

This theorem was developed in 1892 by Aim Vaschy, a French scientist ("Sur les lois de similitude en physique", Annales tlgraphiques, 19:25-28, Vaschy A. (1892)) and Edgar Buckingham, an american physicist, in 1914 ("On physically similar systems; illustrations of the use of dimensional equations", Physical Review, 4(4):345-376). This theorem is also called **PI theorem**. If a physical equation uses n **physical variables**, which depend on k **primary dimensions**, then there is an equivalent equation which uses $n - k$ **dimensionless products**. We can see an example with the function f which depends on n independant parameters q. In Fluid Mechanics, $k = 3$ (if temperature effects are negligible).

The method of this PI theorem is described in three steps:

- It is necessary to select k variables among the n parameters q_i
- we create the $n-k$ groups.
- Finally, exponents values are calculated by writing each quantity with base units and solving the k equation linear system for each exponent set.

John Wheeler, an american physicist said about this theorem: "**Never make a calculation until you know the answer**: make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every puzzle. Courage: no one else needs to know what the guess is. Therefore make it quickly, by instinct. A right guess reinforces this instinct. A wrong guess brings the refreshment of surprise. In either case life as a spacetime expert, however long, is more fun! " (Wheeler, John A. and Edwin F. Taylor. Spacetime Physics, Freeman, 1966. Page 60.)

In order to better understand the method, we are going to study some applications of the Vaschy-Buckingham theorem. The drag force on a sphere can be calculated with this method. Then the dimensionless obtained are the Reynolds number Re_d and the drag coefficient C_D.

The Reynolds number is an important dimensionless number in Fluid Mechanics. It is a measure of the ratio of inertial forces to viscous forces. The concept was introduced by G.G. Stokes in 1851, but the Reynolds number is named after Osbourne Reynolds who popularized its use in 1883. We can see different experiments conducted by Reynolds to study the transition from laminar to turbulent flows.

If we come back to the problem of the drag force on a sphere, we can see the evolution of the drag coefficient, according to the Reynolds number. For each range of Reynolds number, the drag coefficient is expressed by a different law (obtained by experimental measurements).

Now, we can use the same method of Vaschy-Buckingham theorem for a golf ball. What is the difference with the first case ? The surface of a golf ball is not smooth, so the roughness length is added to the problem. With calculations, we see that the drag coefficient depends in this case on the Reynolds number, but also on the relative roughness. The behaviour of the drag coefficient is then changed. This study explains why the golf ball is used to minimize the drag coefficient and improves its trajectory in the air (the golf ball diameter is 42.67mm and its mass is 45.93g).

We have seen with this first section how to improve the analysis of a problem with an easy principle based on the Vaschy-Buckingham theorem. An other advantage of the dimensional analysis is the simplification of balance equations. The dimensional analysis can be used to neglect unimportant terms, in order to simplify a problem but also to obtain information about the solution before solving the problem. For exemple, we can consider a flow around a cylinder. If we write the momentum conservation equations, we obtain terms which characterize the non-stationary acceleration, the convective acceleration, pressure forces, body forces and viscous forces.

Quantities of this equation can be estimated and then compared. We find new dimensionless numbers which characterized the flow, like the **Froude number** (ratio between inertial forces and gravity forces), the **Mach number** (ratio betwen kinetic energy and internal energy), the **Euler number** (ratio between pressure variation and available kinetic energy, for incompressible flows) and the **Strouhal number** (ratio between the non-stationary acceleration and the convective acceleration).

The table includes important dimensionless numbers which can be encountered in flow studies and the case in which they are not negligible.

Now we are going to apply the dimensional analysis to balance equations.

We look at a Newtonian incompressible fluid. By using reduced variables, which are dimensionless parameters, we can rewritten Navier-Stokes equations with new operators. With the calculation of the equation (2), 4 dimensionless numbers appear: Strouhal, Euler, Reynolds and Froude numbers. This new equation can be simplified by chosing a pressure reference. This calculation permit to have new balance equations, using dimensionless numbers.

3 Similarity

Dimensional analysis leads to a partial solution of most problems, but it can not determine complete solutions. Nevertheless, it is possible to reduce the number of variables in a problem. An other application field of the dimensional analysis is similarity problems between a model (a low scale model) and a prototype (at real size).

As Francis Bacon (english scientist) knew, a low scale model is necessary to study a complex problem in order to avoid expensive studies, in many domains (like hydraulics, aerospace, areodynamics...). Overall, all similarity conditions can not be satisfied and we must be satisfied with a partial similarity. It is then necessary to validate the chosen criteria and to offset differences with theoretical or empirical corrections. There are three types of similarity:**geometric**, **kinematic** and **dynamic** similarities.

3.1 Geometric similarity

We can say that two geometrical objects are similar if they both have the same shape. So, all linear **length scales** of one prototype are a fixed ratio of all corresponding length scales of the model and all **angles** are preserved. This similarity seems to be basic but in some cases it is not possible to have a complete geometric similarity!

For instance, hydraulics models are generally distorted with different scale ratios in horizontal and vertical directions. We can understand that a model of the Mont Saint Michel, in order to study the silting of the bay, horizontal and vertical lengths can not be defined by the same scale ratio, in particular because of the scale decrease which can create new phenomena.

3.2 Kinematic similarity

If **length and time scales** are similar between the model and the protoype, it implies that velocities are similar at corresponding points. Then **fluid streamlines** are similar and we have kinematic similarity. We must have geometric similarity to have kinematic similarity.

3.3 Dynamic similarity

If the **forces acting** at corresponding points between the model and the prototype, ewevrywhere in the same ratio, we have the dynamic similarity. This similarity is the most **restrictive**.

To achieve dynamic similarity, **all dimensionless numbers relevant to the flow must be preserved** between the model and the prototype.

It is necessary to have these three similarities to have a complete similarity.

If we study incompressible flows, dimensionless numbers required are the Reynolds number and the Froude number. But, if we have no free surface (for instance with internal pipe flows), then Froude number may be negligible in similarity conditions. On the other hand, if the flow has a free surface with a changing position, Reynolds and Froude numbers have to be preserved. Finally, if surface tension intervenes in studied phenomena, then Weber number must be preserved with Reynolds and Froude numbers (We=kinetic energy/pressure forces (surface tension)).

If we study compressible flows, it is necessary to consider Reynolds ans Mach numbers. But, as it is difficult to preserve Mach and Reynolds numbers for reasonable size of models, if Mach number is small, its effects are theoretically calculated and we choose only Reynolds number to conserve dynamic similarity. For a large Mach number, it must be preserved and Reynolds number must have a realistic value. So we need to perform experiments with a higher pressure or a weaker temperature in order to have valid Reynolds to operate.

Conclusion

We can conclude that dimensional analysis is important to:

- Find physical laws
- simplify problems
- detect relevant parameters
- build a model

And it is very important to master the principle of Vaschy-Buckingham theorem, to know dimensionless numbers and similarity conditions rules.