Dimensional Analysis and Similarity: Solutions

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Studying a cargo-ship with a scale model

1. R depends on: density ρ, viscosity μ, water line length L, velocity v, gravity g.

2. Chosen parameters are: ρ, L_1, v_1. As we have $n = 6$ variables and $k = 3$ units (mass M, length L and time T), we have to find $n - k = 3$ dimensionless numbers.

$[\rho] = ML^{-3}$
$[\mu] = ML^{-1}T^{-1}$
$[L] = L$
$[v] = LT^{-1}$
$[g] = LT^{-2}$
$[R] = MLT^{-2}$

$$\pi_1 = \rho^{\alpha_1} L_1^{\alpha_2} v_1^{\alpha_3} \mu$$
$$\pi_2 = \rho^{\beta_1} L_1^{\beta_2} v_1^{\beta_3} g$$
$$\pi_3 = \rho^{\gamma_1} L_1^{\gamma_2} v_1^{\gamma_3} R$$

$$\pi_1 = \left(ML^{-3}\right)^{\alpha_1} L^{\alpha_2} \left(LT^{-1}\right)^{\alpha_3} ML^{-1}T^{-1}$$
$$\pi_2 = \left(ML^{-3}\right)^{\beta_1} L^{\beta_2} \left(LT^{-1}\right)^{\beta_3} LT^{-2}$$
$$\pi_3 = \left(ML^{-3}\right)^{\gamma_1} L^{\gamma_2} \left(LT^{-1}\right)^{\gamma_3} MLT^{-2}$$

$$\begin{cases} \alpha_1 + 1 = 0 \\ -3\alpha_1 + \alpha_2 + \alpha_3 - 1 = 0 \\ -\alpha_3 - 1 = 0 \end{cases}$$

This system gives: $\begin{cases} \alpha_1 = -1 \\ \alpha_2 = -1 \\ \alpha_3 = -1 \end{cases}$

So, $\pi_1 = \frac{\mu}{\rho L_1 v_1} \simeq \frac{1}{\mathrm{Re}}$

$$\begin{cases} \beta_1 = 0 \\ -3\beta_1 + \beta_2 + \beta_3 + 1 = 0 \\ -\beta_3 - 2 = 0 \end{cases}$$

This system gives: $\begin{cases} \beta_1 = 0 \\ \beta_2 = 1 \\ \beta_3 = -2 \end{cases}$

So, $\pi_2 = \frac{L_1 g}{v_1^2} \simeq \frac{1}{\mathrm{Fr}}$

$\begin{cases} \gamma_1 + 1 = 0 \\ -3\gamma_1 + \gamma_2 + \gamma_3 + 1 = 0 \\ -\gamma_3 - 2 = 0 \end{cases}$

This system gives: $\begin{cases} \gamma_1 = -1 \\ \gamma_2 = -2 \\ \gamma_3 = -2 \end{cases}$

So, $\pi_3 = \frac{R}{\rho L_1^2 v_1^2}$

Finally, we obtain the equation: $\frac{R}{\rho L_1^2 v_1^2} = f\left(\mathrm{Re}, \mathrm{Fr}\right)$

3. As the frictional force $\mathrm{R_f}$ is linked to the fluid viscosity, the group $\frac{\mathrm{R_f}}{\rho L^2 v^2}$ depends only on the **Reynolds number**. On the other hand, the direct resistance $\mathrm{R_d}$ is created due to gravity so $\frac{\mathrm{R_d}}{\rho L^2 v^2}$ depends only on the **Froude number**.

Then: $\frac{\mathrm{R}}{\rho L^2 v^2} = f_1\left(\mathrm{Re}\right) + f_2\left(\mathrm{Fr}\right)$ ("Froude Hypoyhesis").

4. To achieve complete similitude, we must have:

$$\mathrm{Re}_1 = \mathrm{Re}_2 \ \underline{\textbf{and}}\ \mathrm{Fr}_1 = \mathrm{Fr}_2$$

$$\frac{\rho_1 v_1 L_1}{\mu_1} = \frac{\rho_2 v_2 L_2}{\mu_2} \ \underline{\textbf{and}}\ \frac{v_1^2}{gL_1} = \frac{v_2^2}{gL_2}$$

But, $\mu_1 = \mu_2 = \mu$ and $\rho_1 = \rho_2 = \rho$, so we have:

$$v_1 L_1 = v_2 L_2 \ \underline{\textbf{and}}\ \frac{v_1^2}{L_1} = \frac{v_2^2}{L_2}$$

This equation leads to: $\frac{v_1}{v_2} = \frac{L_2}{L_1}$ **and** $\frac{v_1}{v_2} = \sqrt{\frac{L_1}{L_2}}$

This condition is possible only if $\frac{L_2}{L_1} = 1$, so for the real case. It is so **impossible** to have a complete similarity at model scale. As there are a lot of relations to evaluate the frictonal resistance R_f, we will preserve the equality of Froude numbers.

5. The scale is: $\frac{L_1}{L_2} = 30$. The Reynolds number is: $\mathrm{Re}_{L_2} = \frac{\rho v_2 L_2}{\mu}$ with $v_2 = 1.61\,\mathrm{m.s^{-1}}$ and $L_2 = 6\,\mathrm{m}$.

Then: $\mathrm{Re}_{L_2} = 9.66\,10^6$

With the relation $\mathrm{Fr}_1 = \mathrm{Fr}_2$, we obtain:

$$v_1 = v_2\sqrt{\frac{L_1}{L_2}} = 8.82\,\mathrm{m.s^{-1}} \simeq 17.2\,noeuds$$

6. $\mathrm{Re}_{L_2} = 9.66\,10^6$ so $\mathrm{Re}_{L_2} < 10^7$ w have to use the Prandtl relation.

Then, $C_{f_2} = \frac{0.074}{\mathrm{Re}_{L_2}^{0.2}} = 0.003$ and $R_{f_2} = \frac{1}{2} C_{f_2} \rho S_2 v_2^2 = 15.55\,\mathrm{N}$

7. $R_2 = 20\,\mathrm{N}$ and $R_{f_2} = 15.55\,\mathrm{N}$ so $R_{d_2} = R_2 - R_{f_2} = 4.45\,\mathrm{N}$

We can estimate the direct resistance of the cargo-ship because similarity conditions of Froude numbers are respected. Then:

$$\frac{R_{d_1}}{\rho L_1^2 v_1^2} = \frac{R_{d_2}}{\rho L_2^2 v_2^2}$$

So, we obtain: $R_{d_1} = R_{d_2} \frac{L_1^2 v_1^2}{L_2^2 v_2^2} = 1.20\,10^5\,\mathrm{N}$

To calculate R_1, we must have R_{f_1}. As Reynolds numbers are not preserved, we can not deduce R_{f_1} from the model measurement. But we can calculate R_{f_1} from relations given in the question 6.:

$$\mathrm{Re}_{L_1} = \frac{\rho v_1 L_1}{\mu} = 15.88\,10^9$$

With this result, we use the Prandtl-Schlichting relation:

$$C_{f_1} = \frac{0.455}{(\log \mathrm{Re})^{2.584}} = 1.70\,10^4$$

So, $R_{f_1} = \frac{1}{2} C_{f_1} \rho S_1 v_1^2 = 2.38\,10^4\,\mathrm{N}$.

Then, we obtain: $R_1 = R_{f_1} + R_{d_1} = 1.438\,10^5\,\mathrm{N}$

8. The power requisites to propel the cargo-ship is given by: $\mathcal{P}_1 = R_1 v_1 = 1.27\,\mathrm{MW}$

The atomic explosion of 1945

1. The energy released E depends on:

- Time t
- Density ρ
- Radius R
- Ratio of specific heats γ

Then, there are $n = 5$ variables.

2. Units of each variable are:

- $[E] = ML^2T^{-2}$
- $[\gamma] = 1$
- $[t] = T$
- $[R] = L$
- $[\rho] = ML^{-3}$

So $k = 3$ and $n - k = 2$: there are two dimensionless numbers. If chosen variables are t, R and ρ, dimensionless products are written as:

$$\pi_1 = t^{\alpha_1} R^{\alpha_2} \rho^{\alpha_3} E \tag{1}$$

$$\pi_2 = t^{\beta_1} R^{\beta_2} \rho^{\beta_3} \gamma \tag{2}$$

Then: $[\pi_1] = T^{\alpha_1} L^{\alpha_2} \left(ML^{-3}\right)^{\alpha_3} ML^2T^{-2}$ and $\pi_2 = T^{\beta_1} L^{\beta_2} \left(ML^{-3}\right)^{\beta_3}$

The two systems of equations to resolve are:

$$\begin{cases} \alpha_1 - 2 = 0 \\ \alpha_2 - 3\alpha_3 + 2 = 0 \\ \alpha_3 + 1 = 0 \end{cases} \tag{3}$$

$$\begin{cases} \beta_1 = 0 \\ \beta_2 - 3\beta_3 = 0 \\ \beta_3 = 0 \end{cases} \tag{4}$$

Solving these two systems (3) and (4) gives:

$$\begin{cases} \alpha_1 = 2 \\ \alpha_2 = -5 \\ \alpha_3 = -1 \end{cases} \tag{5}$$

$$\begin{cases} \beta_1 = 0 \\ \beta_2 = 0 \\ \beta_3 = 0 \end{cases} \tag{6}$$

Then, dimensionless numbers are: $\pi_1 = \frac{t^2 E}{R^5 \rho}$ and $\pi_2 = \gamma$.

3. The released energy can then be expressed as: $E = f(\gamma)\frac{\rho R^5}{t^2}$.

4. See Figure 1 in the second Paper.

5. For $t = 0.062\,\mathrm{s}$, $R = 185\,\mathrm{s}$. As $\rho = 1.2\,\mathrm{kg.m^{-3}}$ for 20°C, we obtain:

$$E = 6.76\,10^{13}\,\mathrm{J} = 0.016\,\mathrm{Mt\ of\ TNT}.$$