Sensors and measurements in fluid mechanics: solutions

A. Danlos[a], F. Ravelet[a]
[a] *Arts et Metiers ParisTech, DynFluid,*
151 boulevard de l'Hôpital, 75013 Paris, France.
contact: florent.ravelet@ensam.eu

January 24, 2014

1 Flow rate measurement with inclined venturi

1. The studied flow is stationary and presents no viscous forces. It is then possible to used the Bernoulli's theorem. If we consider a streamline between sections A_0 and A_1, we have:

$$p_0 + \rho g z_0 + \rho \frac{v_0^2}{2} = p_1 + \rho g z_1 + \rho \frac{v_1^2}{2} \tag{1}$$

where p_0, v_0, p_1 and v_1 are pressures and velocities respectively in A_0 and A_1.

Since the flow is unidimensional in A_0 and A_1, velocities v_0 and v_1 are uniform on each sections.

2. Hydrostatics laws are applied in the U-tube and its connections, because pressure tappings are perpendicular to the flow (figure 1).

We obtain:

$$\begin{aligned} p_{ref} &= p_0 + \rho g \left(z_0 - z_2\right) \\ p_{ref} &= p_1 + \rho g \left(z - 1 - z_3\right) + \rho_m g \left(z_3 - z_2\right) \end{aligned}$$

where p_{ref} is the pressure in the line indicated as the pressures reference.

3. With a combination of these two equations, we obtain:

$$p_0 - p_1 = \rho g \left(z_1 - z_3 - z_0 + z_2\right) + \rho_m g \left(z_3 - z_2\right)$$

$$\left(p_0 + \rho g z_0\right) - \left(p_1 + \rho g z_1\right) = \left(\rho_m - \rho\right) g \Delta h \tag{2}$$

4. In order to obtain a relation between velocities in section A_0 and A_1, we substitute the equation (2) in the equation (2):

$$v_1^2 - v_0^2 = 2 \frac{\rho_m - \rho}{\rho} g \Delta h \tag{3}$$

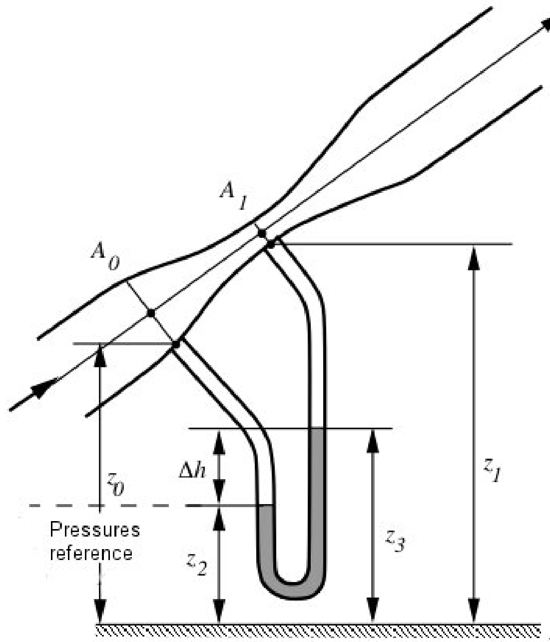

Figure 1: Pressures reference

The mass conservation (or flow rate conservation because the flow is incompressible) between sections A_0 and A_1 is then :

$$\frac{\dot{m}}{\rho} = q_v = \frac{\pi D_0 2}{4} v_0 = \frac{\pi D_1 2}{4} v_1 \tag{4}$$

Using the equation (4) and the equation (3), we can determine the volumetric flow rate:

$$q_v = \frac{\pi}{4} \left(\frac{1}{\frac{1}{D_1^4} - \frac{1}{D_0^4}} \right)^{1/2} \left(2 \frac{\rho_m - \rho}{\rho} g \Delta h \right)^{1/2}$$

We can see that this relation does not depend on the venturi inclination.

5. The value of the volumetric flow rate is then: $q_v = 0.0156\,\mathrm{m}^3.\mathrm{s}^{-1}$

2 Couette viscometer

1. The height of oil $h = 20\,\mathrm{cm}$ is large compared to the distance $\delta = 1\,\mathrm{mm}$ between the two cylinders: $\frac{\delta}{h} \ll 1$.

We can then neglect the edge effects and especially the flow in the bottom of the viscometer.The hydrostatic pressure gradient induced by the gravity force produces no vertical flow, and the tangential pressure gradient is zero. So we can consider that the problem study a flow between two cylinders with infinite heights, where the velocity v has no vertical component and does not depend on the coordinate z. In a rotating system we can expressed the velocity $\vec{v}$:

$$\vec{v} = u_r(r,\theta)\,\vec{e_r} + u_\theta(r,\theta)\,\vec{e_\theta}$$

The flow in a stationary regime is established. Then it does not depend on θ. The distance between the two walls $a = R_2 - R_1 = 0.02\,\text{cm}$ is small compared to the average radius of curvature $R = (R_1 + R_2)/2 = 5.01\,\text{cm}$. We have: $a/R \ll 1$, so the curvature effects can be neglegted.

As the centrifugal force is still low, $u_r = 0$ and only the rotation of the external cylinder induces a flow in blocks parallel to the plane Oxy by driving the fluid layers:

$$\vec{v} = u_\theta(r)\,\vec{e_\theta}$$

As $a/R \ll 1$, we can use $y = r - R_1$ and the flow between the two cylinders is considered as a unidimensional flow between two infinite planes separated by $a = 0.02\,\text{cm}$. One of these planes is motionless and the other one has a motion with a uniform velocity V (figure 2

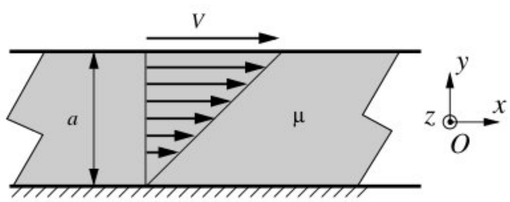

Figure 2: The studied flow of oil between the two cylinders walls

The external cylinder rotates at $N = 90\,\text{rpm}$. The movable plane has a velocity V:

$$V = \omega R_2 = \frac{2\pi N}{60} R_2 = 0.473\,\text{m.s}^{-1}$$

2. The velocity field is: $u = u(y)$. The acceleration $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\partial \vec{v}}{\partial t} + \vec{v}.\vec{\nabla}\vec{v}$ is then zero. Without pressure gradient and volume force in the direction of the x axis, the projection of the Navier-Stokes equations can give an expression for the tangential stress τ_{xy}:

$$\frac{\partial \tau_{xy}}{\partial y} = 0$$

As the fluid is newtonian: $\tau_{xy} = \mu \partial u/\partial y$. Then, we obtain: $\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0$.

This equation can be integrated in the shape: $u = c_1 y + c_2$. The velocity profile is then linear between the two walls (figure 2).

The attached condition of the fluid to the fixed wall $u(0) = 0$ and to the movable wall $u(a) = V$ allows to determine the exact equation:

$$u(y) = \frac{V}{a} y$$

3. The stress τ_0 applied by the fluid fluid on the side wall of the inner cylinder is given by:

$$\tau_0 = \tau_{xy}(y=0) = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{V}{a}$$

4. With the rotational symmetry, the resultant of the forces applied by the oil in motion on the inner cylinder is zero, but the fluid applied a torque C on the torque cable. The force applied by the flow on the fixed plane is given by the resultant of the wall stress $F = \tau_0 S_1$. In order to calculate the torque applied on the torque cable, we determine:

$$C = FR_1 = \tau_0 S_1 R_1 = \mu \frac{V}{a} S_1 R_1$$

where $S_1 = 2\pi R_1 h$ is the external surface of the inner cylinder immersed in the fluid.

5. As the flow is at rest for distances $r < R_1$, we can finally determine the viscosity μ:

$$\mu = \frac{Ca}{R_1 S_1 V}$$

Calculation of the viscosity for given data induces: $\mu = 1.48 kg.m^{-1}.s^{-1}$

3 Falling drop measurement

1. Forces acting on the drop:

- its apparent weight: $\mathcal{P} = (\rho_d - \rho_g) V_g$, directed downward
- Surface tension force: $T = 2\pi r\gamma$, directed upward
- overpressure: $\Delta p = \gamma \left(\frac{1}{r} - \frac{1}{R}\right)$, the force $F_L = \Delta p \pi r^2$ is directed downward.

2.

$$(\rho_d - \rho_g) v_g - 2\pi r\gamma + \gamma \left(\frac{1}{r} - \frac{1}{R}\right) \pi r^2 = 0$$
$$(\rho_d - \rho_g) v_g = \pi r\gamma \left(1 + \frac{r}{R}\right)$$

Then, we have: $\gamma = \frac{V_g}{\pi r} \frac{(\rho_d - \rho_g)}{1 + \frac{r}{R}}$
3. As $R >> r$, we can assume that $\frac{r}{R}$ is negligible, and the radius r is also the external radius of the capillary tube. Then:

$$(\rho_d - \rho_g) V_g = \pi a \gamma$$

We know that: $m = \rho_d V$, so $\gamma = \frac{\frac{mg}{\rho_d}(\rho_d - \rho_g)}{\pi a}$

$$\gamma = \frac{mg\left(1 - \frac{\rho_g}{\rho_d}\right)}{\pi a}$$

4. Surrounding gas is air so: $\rho_g << \rho_d$ and $\frac{\rho_g}{\rho_d}$ is negligible.
We have then: $\gamma = \frac{mg}{\pi a}$

$$\rho_{oil} = d\rho_{water} = 977.06\,\mathrm{kg.m^{-3}}$$
$$\text{with } d = 0.98 \text{ for } T = 22^\circ\,\mathrm{C} \text{ and } \rho_{water} = 997\,\mathrm{kg.m^{-3}}$$

As we have $a = 0.8\,\mathrm{mm}$ and $M_{total} = 1.094\,\mathrm{g}$, we have:

$$m = \frac{M_{total}}{200} = 0.00547\,\mathrm{g}$$

Then: $V = \frac{m}{\rho_{oil}} = 6\,10^{-9}\,\mathrm{m^3} = 0.006\,\mathrm{mL}$
We can calculate: $\frac{a}{V^{1/3}} = 0.44026$, so $f = 1.3258$
We obtain: $\gamma_{oil} = \frac{mg}{\pi f a} = 16.11\,10^{-3}\,\mathrm{N.m^{-1}}$